ON THE PROBLEM OF POSITION DETERMINATION OF A NONLINEAR SYSTEM IN PHASE SPACE

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In monograph [1] a method is presented for position determination of a nonlinear system in phase space. One of possible approximate methods of solution of nonlinear integral equations obtained in [1] is given below.

Equations of motion of the control system are presented in the following form

$$z_j^{\cdot} + \sum_{k=1}^{r} a_{jk}(t)\, z_k = X_j(t) + \psi_j(z_1, \ldots, z_r, t) \qquad (j = 1, \ldots, r) \tag{1}$$

Here z_j are phase coordinates of the system, $X_j(t)$ are known external forces, $\psi_j(z_1, \ldots, z_r, t)$ are nonlinear functions, continuous with respect to all their arguments in some region and satisfying Lipschitz conditions with respect to $z_1, \ldots, z_r$ in this region.

In [1] a method is examined for determination of initial values of all phase coordinates of the system from observable increments of one or several coordinates. Let one coordinate z_s be accessible to measurement. Designating through $S(t)$ the deviation of coordinate z_s with respect to some arbitrary, but fixed origin of reference, we obtain

$$z_s(t_j) - z_s(t_0) = S(t_j) - S(t_0) = L_j \qquad (j = 1, \ldots, r) \tag{2}$$

Here $t_0, t_1, \ldots, t_r$ are some instants of time.

The following system of integral equations is equivalent to the system (1):

$$z_j(t) = \sum_{k=1}^{r} N_{jk}(t, t_0)\, z_k(t_0) + g_j(t) + I_j(t) \qquad (j = 1, \ldots, r)$$

$$g_j(t) = \int_{t_0}^{t} \sum_{k=1}^{r} N_{jk}(t, \tau)\, X_k(\tau)\, d\tau, \qquad I_j(t) = \int_{t_0}^{t} \sum_{k=1}^{r} N_{jk}(t, \tau)\, \psi_k[z_1(\tau), \ldots, z_r(\tau), \tau]\, d\tau \tag{3}$$

Here $N_{jk}(t, \tau)$ are elements of matrix weight function for the system

$$z_j^{\cdot} + \sum_{k=1}^{r} a_{jk}(t)\, z_k = 0 \qquad (j = 1, \ldots, r)$$

Substituting Expression (3) for $z_s(t_j)$ into (2) we obtain

$$\sum_{k=1}^{r} b_{jk} z_k(t_0) = L_j - g_s(t_j) - I_s(t_j), \quad b_{jk} = N_{sk}(t_j, t_0) - N_{sk}(t_0, t_0) \quad (j = 1, \ldots, r) \tag{4}$$

Solving system (4) with respect to $z_k(t_0)$, we find

$$z_k(t_0) = \gamma_k - \sum_{\nu=1}^{r} m_{k\nu} I_s(t_\nu), \qquad \gamma_k = \sum_{\nu=1}^{r} m_{k\nu} [L_\nu - g_s(t_\nu)] \tag{5}$$

Here $m = \| m_{jk} \|$ is a matrix which is inverse of the matrix $b = \| b_{jk} \|$. Substituting into (3) instead of $z_k(t_0)$ their expression from (5), we obtain a system of integral equations

$$z_j(t) = \sum_{k=1}^{r} N_{jk}(t, t_0)\gamma_k + g_j(t) - \sum_{k=1}^{r} R_{jk}(t) I_s(t_k) + I_j(t) \qquad (j = 1, \ldots, r)$$
$$R_{jk}(t) = \sum_{\nu=1}^{r} N_{j\nu}(t, t_0) m_{\nu k} \tag{6}$$

Here $g_j(t)$ and $I_j(t)$ are determined by Expressions (3).

In the case when all nonlinear functions $\psi_j(z_1, \ldots, z_r, t)$ depend only on one coordinate, i.e.

$$\psi_j(z_1, \ldots, z_r, t) = \psi_j(z_s, t) \qquad (j = 1, \ldots, r) \tag{7}$$

we will have one integral equation

$$z_s(t) = \sum_{k=1}^{r} N_{sk}(t, t_0)\gamma_k + g_s(t) - \sum_{k=1}^{r} R_{sk}(t) I_s(t_k) + I_s(t) \tag{8}$$

where according to (3) and (7)

$$I_s(t) = \int_{t_0}^{t} \sum_{k=1}^{r} N_{sk}(t, \tau)\psi_k[z_s(\tau), \tau]\, d\tau \tag{9}$$

Further, a solution is required for the following system of linear differential equations

$$y_j^{\bullet} + \sum_{k=1}^{r} a_{jk}(t) y_k = X_j(t) \qquad (j = 1, \ldots, r)$$

for initial conditions $y_j(t_0) = \gamma_j$. This solution has the form

$$y_j(t) = \sum_{k=1}^{r} N_{jk}(t, t_0)\gamma_k + g_j(t)$$

where $g_j(t)$ are determined from Equations (3). Taking into consideration that matrix m is the inverse of matrix b, we can obtain, in agreement with (4) and (5) the relationships

$$y_s(t_j) - y_s(t_0) = \sum_{k=1}^{r} b_{jk}\gamma_k + g_s(t_j) = \sum_{k, \nu=1}^{r} b_{jk} m_{k\nu} [L_\nu - g_s(t_\nu)] + g_s(t_j) = L_j \tag{10}$$

Examining the differences

$$u(t) = z_s(t) - z_s(t_0), \qquad v(t) = y_s(t) - y_s(t_0) \tag{11}$$

we will have according to (2) and (10)

$$u(t_j) = v(t_j) = L_j, \quad z_s(t_j) = v(t_j) + z_s(t_0) \qquad (j = 1, \ldots, r) \tag{12}$$

Taking into consideration relationship (12), we can solve the nonlinear integral equation (8) by an approximate method. Namely, taking into account (12) we will assume that

$$z_s(t) \approx v(t) + z_s(t_0) \quad \text{for} \quad t \in [t_0, t_r] \tag{13}$$

and according to (9)

$$I_s(t) \approx c[z_s(t_0), t] = \int_{t_0}^{t} \sum_{k=1}^{r} N_{sk}(t, \tau) \psi_k [v(\tau) + z_s(t_0), \tau] \, d\tau \tag{14}$$

This approximation becomes more accurate the smaller the interval $[t_0, t_r]$.

Approximate values $z_s(t)$ and $I_s(t)$ according to (13) and (14) are substituted into (8). Then, assuming that $t = t^* \in [t_0, t_r]$ and substituting $z_s(t_0)$ by Z, we obtain for Z the following equation:

$$Z = \gamma_s - \sum_{k=1}^{r} R_{sk}(t^*) \, c(Z, t_k) + c(Z, t^*) \tag{15}$$

It follows from (15) that the error $\varepsilon(t^*) = Z(t^*) - z_s(t_0)$ will be a function of t^*. We will prove that at all points $t^* = t_j$ $(j = 0, 1, \ldots, r)$ this error has one and the same value

$$\varepsilon(t_j) = Z(t_j) - z_s(t_0) = \varepsilon \qquad (j = 0, 1, \ldots, r)$$

For this we will show that all $r + 1$ relationships which are obtained from (15) for $t^* = t_j$ are equivalent to one and the same relationship

$$Z = \gamma_s - \sum_{k=1}^{r} m_{sk} \, c(Z, t_k) \tag{16}$$

Correctness of the last statement follows from Equation

$$R_{sk}(t_j) = \begin{cases} m_{sk} + 1 & j = k \\ m_{sk} & j \neq k \end{cases} \qquad \begin{pmatrix} j = 0, 1, \ldots, r \\ k = 1, \ldots, r \end{pmatrix} \tag{17}$$

which can be obtaindd thus: substituting into (2) Expression (8) for $z_s(t_j)$ we have

$$L_j = \sum_{k=1}^{r} b_{jk}\gamma_k - \sum_{k=1}^{r} \rho_{jk} I_s(t_k) + I_s(t_j) + g_s(t_j) \qquad (j = 1, \ldots, r)$$
$$\rho_{jk} = R_{sk}(t_j) - R_{sk}(t_0) \tag{18}$$

Introducing matrices

$$I = \begin{Vmatrix} I_s(t_1) \\ \vdots \\ I_s(t_r) \end{Vmatrix}, \quad g = \begin{Vmatrix} g_s(t_1) \\ \vdots \\ g_s(t_r) \end{Vmatrix}, \quad L = \begin{Vmatrix} L_1 \\ \vdots \\ L_r \end{Vmatrix}, \quad \gamma = \begin{Vmatrix} \gamma_1 \\ \vdots \\ \gamma_r \end{Vmatrix}, \quad \begin{matrix} \rho = \| \rho_{jk} \| \\ b = \| b_{jk} \| \\ m = \| m_{jk} \| \end{matrix} \quad (j, k = 1, \ldots, r)$$

relationships (18) and (5) are presented in the matrix form

$$L = b\gamma - \rho I + I + g, \qquad \gamma = m(L - g) \tag{19}$$

Taking into account that $m = b^{-1}$ it is easy to see from (19) that $(-\rho + E)I = 0$. Consequently, $\rho = E$, where E is a unit matrix of the order of $(r \times r)$. Now justification of relationships (17) follows from (18) and (6). It follows from the material presented that in order to obtain an approximate value $z_s(t_0)$ it is expedient to use Equation (16) which is obtained from (15) for $t^* = t_j (j = 0, 1, \ldots, r)$.

After $Z \approx z_s(t_0)$ is found from Equation (16), initial values of remaining coordinates $z_j(t_0)$ are detrmined according to (5) and (14) from Equations

$$z_j(t_0) \approx \gamma_j - \sum_{k=1}^{r} m_{jk}\, c(Z, t_k) \qquad (j = 1, \ldots, r;\ \ j \neq s) \tag{20}$$

If in Equations (1) nonlinear functions are contained which depend on several coordinates z_{p_k}, then for application of the method examined above it is necessary to have in the system corresponding increments

$$S_{p_k}(t_j) - S_{p_k}(t_0)$$

with respect to all these coordinates, because only in this case is it possible to utilize approximations analogous to (13).

As an example let us investigate the problem of determination of initial values of generalized coordinates of the gyrocompass with a nonlinear restoring force examined in [1]. Equations of motion of the system have the form

$$z_j^{\bullet} + a_{j1}z_1 + a_{j2}z_2 + a_{j3}z_3 = \psi_j(z_1) \qquad (j = 1, 2, 3) \tag{21}$$

$$z_1 = \alpha, \qquad z_2 = \beta - \frac{HU\sin\varphi}{\rho lP}, \qquad z_3 = \vartheta + \frac{HU\sin\varphi}{\rho lP}$$

$$k^2 = \frac{lPU\cos\varphi}{H}, \qquad \psi_2(z_1) = -\zeta z_1^3, \qquad \psi_1(z_1) = \psi_3(z_1) = 0$$

The matrix of coefficients a_{jk} has the form

$$\begin{Vmatrix} 0, & -k^2/U\cos\varphi, & -k^2(1-\rho)/U\cos\varphi \\ U\cos\varphi, & 0, & 0 \\ 0, & F, & F \end{Vmatrix} \tag{22}$$

Here α is the angle of deflection of the gyrocompass in the azimuth, β is the angle of elevation of the northern diameter of the gyrosphere above the horizontal plane, ϑ is the angle of inclination of the mirror surface of the liquid of the hydraulic damper above the equator plane of the gyrosphere, H is the moment of momentum, lP is the static moment of the sensitive element, ρ, F are parameters of the hydraulic damper, $\psi_2(z_1)$ is the nonlinear restoring force, U is the angular velocity of rotation of the Earth sphere, φ is the latitude at the point of observation. As coordinate with respect to which measurments of increments L_j are made we select deflection of the gyrocompass in the azimuth with respect to some arbitrary, but fixed origin of reference

$$L_j = S(t_j) - S(t_0) = z_1(t_j) - z_1(t_0) \qquad (j = 1, 2, 3) \tag{23}$$

where t_0, t_1, t_2, t_3 are some instants of time. Equation (16) takes the form

$$Z = \gamma_1 - \sum_{k=1}^{3} m_{1k}\, c(Z, t_k), \qquad \gamma_j = \sum_{k=1}^{3} m_{jk} L_k$$

$$\| m_{jk} \| = \| b_{jk} \|^{-1}, \quad b_{jk} = N_{1k}(t_j, t_0) - N_{1k}(t_0, t_0) \quad (j = 1, 2, 3) \tag{24}$$

$$c(Z, t) = \int_{t_0}^{t} N_{12}(t, \tau)\, \psi_2[v(\tau) + Z]\, d\tau, \qquad v(t) = \sum_{k=1}^{3} N_{1k}(t, t_0)\, \gamma_k - \gamma_1$$

Here $N_{jk}(t, \tau)$ are elements of matrix weight function of system (21) for $\psi_2(z_1) = 0$.

Taking into account the form of the nonlinear function $\psi_2(z_1)$ we present Equation (24) in the form

$$a_0Z^3 + 3a_1Z^2 + (3a_2 - 1)Z + a_3 + \gamma_1 = 0$$
$$a_\nu = \zeta \sum_{k=1}^{3} m_{1k}\omega_{k\nu}, \quad \omega_{k\nu} = \int_{t_0}^{t_k} N_{12}(t_k, \tau) v^\nu(\tau)\, d\tau \qquad (\nu = 0, 1, 2, 3) \tag{25}$$

After determination of $Z \approx z_1(t_0)$ from Equation (25), the quantities $z_2(t_0)$ and $z_3(t_0)$ are determined from Equation (20) which for the present problem takes the form

$$z_j(t_0) \approx \gamma_j + \zeta \sum_{k=1}^{3} m_{jk}(\omega_{k0}Z^3 + 3\omega_{k1}Z^2 + 3\omega_{k2}Z + \omega_{k3}) \qquad (j = 2, 3) \tag{26}$$

Quantities $N_{1k}(t, t_0)$ and $\omega_{k\nu}$ can be calculated on a computer by means of integration for various initial conditions using the right-hand parts of the following system of differential equations (27)

$$y_j^{\bullet} + \sum_{k=1}^{3} a_{jk}y_k = 0, \quad x_{j\nu}^{\bullet} + \sum_{k=1}^{3} a_{jk}x_{k\nu} = \varepsilon_j (y_1 - \gamma_1)^\nu \quad \begin{pmatrix} \varepsilon_1 = 0 \\ \varepsilon_2 = 1 \\ \varepsilon_3 = 0 \end{pmatrix} \quad \begin{pmatrix} j = 1, 2, 3 \\ \nu = 0, 1, 2, 3 \end{pmatrix}$$

General solution of system (27) has the form (28)

$$y_j(t) = \sum_{k=1}^{3} N_{jk}(t, t_0) y_k(t_0), \qquad x_{j\nu} = \sum_{k=1}^{3} N_{jk}(t, t_0) x_{k\nu}(t_0) + \int_{t_0}^{t} N_{j2}(t, \tau)[y_1(\tau) - \gamma_1]^\nu d\tau$$

From (25), (24) and (28) result the following relationships

$$N_{1k}(t, t_0) = y_1(t) \quad \text{for } y_j(t_0) = 0, \quad j \neq k, \quad y_k(t_0) = 1$$
$$\omega_{k\nu} = x_{1\nu}(t_k) \quad \text{for } y_j(t_0) = \gamma_j, \quad x_{j\nu}(t_0) = 0 \tag{29}$$

Calculations were carried out for the following values of parameters of the system:

$$k^2 = 1.53921 \cdot 10^{-6}\,\text{sec}^{-2}, \quad \rho = 0.38, \quad F = 1.5 \cdot 10^{-3}\,\text{sec}^{-2}, \quad \zeta = 0.4 \cdot 10^{-3}\,\text{sec}^{-1}$$
$$U\cos\varphi = 4.11368 \cdot 10^{-5}\,\text{sec}^{-1}, \quad t_0 = 0, \quad t_1 = 150\,\text{sec}, \quad t_2 = 300\,\text{sec}, \quad t_3 = 450\,\text{sec}$$

Calculated L_1, determined according to (23), were found by means of integration of system (21) for initial conditions $z_1(0) = 0.3$, $z_2(0) = 0.004$, $z_3(0) = 0.004$.

Coefficients a_ν were computed in accordance with (25) and (29) by means of integration of system (27) on a computer. In this process Equation (25) turned out to be the following:

$$9.723656Z^3 + 0.212817Z^2 + 0.998616Z - 0.581265 = 0 \tag{30}$$

Equation (30) has the solution: $Z = 0.299997$; the other two roots are complex.

Approximate initial values of phase coordinates obtained as a result of solving Equation (30) and according to Equations (26) will be the following:

$$z_1(0) \approx 0.299997, \quad z_2(0) \approx 0.004001, \quad z_3(0) \approx 0.003998 \tag{31}$$

Values (31) are sufficiently close to assumed initial values.

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